The universal HKR (Moulinos, Robalo, Toën)

I) Review of HH

: All rings are commutative, fix a ring k

$$\mathrm{CAlg}_k^{an} = \{\text{animated comm } k\text{-algebras}\} = \mathrm{SCR}_k$$

<u>Def</u>: For a map $A \to B$ in CAlg_k^{an}, set $HH(B/A) = B \underset{B \otimes_A B}{\otimes} B \in \mathrm{CAlg}_k^{an}$

<u>ex</u>: 1) $HH_0(B/A) = B$, $HH_1(B/A) = I_\Delta / I_\Delta^2 \cong \Omega^1_{B/A}$ $\left(\begin{array}{l} A \text{ discrete} \\ B \text{ discrete} \end{array}\right)$

2) $A \to B$ smooth map of ord rings

<u>Thm (HKR)</u>: $\Omega^i_{B/A} \xrightarrow{\sim} HH_i(B/A)$

<u>Pf</u>: do a Koszul calculation (exercise)

Rmk: 1) $HH(-/A) : CAlg_A^{an} \longrightarrow CAlg_A^{an}$ is left-Kan extended from poly A-algebras

$\Rightarrow \exists$ a natural filtration Fil^i_{HKR} on $HH(B/A)$ s.t

$$gr^i_{HKR} HH(B/A) \cong \Lambda^i L_{B/A}[i]$$

Goal of the seminar: construct this filtration more conceptually

2) $HH(B/A) = B \underset{A}{\otimes} S^1$ in $CAlg_A^{an}$ $\left(\text{i.e. } Map_{CAlg_A^{an}}(HH(B/A), -) \cong Map_{Spaces}(S^1, Map_{CAlg_A^{an}}(B,-)) \right)$

Pf: $S^1 = \operatorname{colim} \begin{pmatrix} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{pmatrix}$

$\therefore B \underset{A}{\otimes} S^1 = \operatorname{colim} \begin{pmatrix} B \underset{A}{\otimes} B & \longrightarrow & B \\ \downarrow & & \\ B & & \end{pmatrix} = B \underset{B \underset{A}{\otimes} B}{\otimes} B =: HH(B/A)$

Upshot: $HH(B/A) \in \mathrm{CAlg}_A^{an}$ has an S^1 action

$\rightsquigarrow$ $\sigma \in H_1(S^1)$ induces $HH_*(B/A) \xrightarrow{B} HH_{*+1}(B/A)$

Exercise: If $A \to B$ smooth, then $B = d_{dR}$ under $HH_*(B/A) \cong \Omega^*_{B/A}$

(3) Using (2), can show

Thm (Ben-Zvi + Francis - Nadler): $A \to B$ in CAlg_A^{an}, $X = \mathrm{Spec}(B)$

$$LX := \underline{\mathrm{Map}}(S^1, X) \in \mathrm{dSt}_A = \mathrm{PShv}_{\mathrm{spaces}}((\mathrm{CAlg}_A^{an})^{op})$$

$\rightsquigarrow$ $R\Gamma(LX, \mathcal{O}_X) \cong HH(B/A)$

$$\underline{\mathrm{Map}}\left(\mathrm{colim}\left(\begin{array}{ccc} \cdot & \to & \cdot \\ \downarrow & & \\ \cdot & & \end{array}\right), X\right) = \varprojlim \left(\begin{array}{ccc} & & X \\ & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}\right) = X \times_{X \times X} X$$

<u>Goals</u>: (i) Realize $\underline{\mathrm{Map}}(S^1, X)$ as $\underline{\mathrm{Map}}(\mathrm{Aff}(S^1), X)$

where $\mathrm{Aff}(S^1)$ = alg stack / k

(2) Construct a filtration on $\mathrm{Aff}(S^1)$ inducing the HKR filtration

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II) <u>Affinization</u>: k comm ring, $\mathrm{St}_k = \mathrm{PShv}_{\text{spaces}}(\mathrm{CAlg}_k^{cl})$

$$\mathrm{coSCR}_k = \{\text{cosimp comm } k\text{-algebras}\} \leftarrow \text{as } \infty\text{-category}$$

<u>Obs</u>: $\mathrm{Spec}^{\Delta} : \mathrm{coSCR}_k^{op} \longrightarrow \mathrm{St}_k$

$$R^{\bullet} \longmapsto \left(B \longmapsto \mathrm{Map}_{\mathrm{coSCR}_k}(R^{\bullet}, B) \right)$$

Thm (Toën): Spec^{Δ} is fully faithful with left-adjoint given by

$$X \longmapsto R\Gamma(X, \mathcal{O}_X)$$

Def: ① For any $X \in \mathrm{St}_k$, write $X \longrightarrow \mathrm{Aff}(X)$

$$\mathrm{Aff}(X) = \mathrm{Spec}^{\Delta}(R\Gamma(X, \mathcal{O}_X))$$

② X is <u>affine</u> if $X \xrightarrow{\sim} \mathrm{Aff}(X)$

Ex:

1) X affine scheme $\Rightarrow$ X affine stack (and in fact any quasi-affine scheme)

2) $K(\mathbb{G}_a, n)$ is affine

$R\Gamma(K(\mathbb{G}_a, n), \mathcal{O}) =$ free object in coSCR_k on a deg n class

3) If $\mathrm{char}(k)=0$, then $\mathrm{Aff}(S^1) \simeq B\mathbb{G}_a = K(\mathbb{G}_a,1)$

(Pf: $C^*(S^1,k) \simeq C^*(B\mathbb{G}_a,\mathcal{O})$)

<u>Thm (this paper)</u>: $\underline{\mathrm{Map}}(S^1,X) \simeq \underline{\mathrm{Map}}(\mathrm{Aff}(S^1),X)$

in dSt_k for X affine derived k-scheme

<u>III) Filtrations</u>:

<u>Thm (Simpson, Moulinos)</u>: k comm ring

1) $$\begin{array}{ccc} DF(k) & \underset{\text{Rees}}{\simeq} & QCoh(\mathbb{A}^1/\mathbb{G}_m) \\ \parallel & & \\ Fun(\mathbb{Z}, D(k)) & & \end{array}$$

2) Under (*), some natural operations match up:

$$\begin{array}{ccc} DF(k) & & QCoh(\mathbb{A}^1/\mathbb{G}_m) \\ \big\downarrow \text{forget} & \Longleftrightarrow & \big\downarrow \text{restriction} \\ D(k) & & QCoh(pt) = QCoh(\mathbb{G}_m/\mathbb{G}_m) \end{array}$$

$$\begin{array}{ccc} DF(k) & & QCoh(\mathbb{A}^1/\mathbb{G}_m) \\ \big\downarrow gr^{\bullet} & \Longleftrightarrow & \big\downarrow \text{restriction} \\ D(k)^{\mathbb{Z}\text{-}gr} & & QCoh(B\mathbb{G}_m) \end{array}$$

__Def__: A __filtration on a stack__ X is a stack $\mathcal{X} \xrightarrow{f} \mathbb{A}^1/\mathbb{G}_m$ + $f^{-1}(1) \cong X$

$$\rightsquigarrow \quad X^{gr} = f^{-1}(0) \longrightarrow B\mathbb{G}_m$$

IV) Some group schemes, Fix a prime p

Recall: $W(-) \equiv$ p-typical Witt vector functor

$$\equiv \varprojlim_n W_n(-)$$

isomorphic to $\mathbb{A}^{n+1}$ as scheme

Have $\underset{\overset{\shortparallel}{F}}{\text{Frob}} : W(-) \longrightarrow W(-)$ Frobenius map

Def: $\text{Fix} = \ker\left(W(-) \xrightarrow{F-id} W(-)\right)$

$\text{Ker} = \ker\left(W(-) \xrightarrow{F} W(-)\right)$

both group schemes

Easy Lemma: If $\text{char}(k)=0$ then $\text{Fix} \equiv \text{Ker} \equiv \mathbb{G}_a$

Thm: $\exists$ a filtration on $B\,\mathrm{Fix}$ with assoc gr $B\,\mathrm{Ker}$

idea: Consider a 1-parameter family of endomorphisms

$$W(-) \xrightarrow{F - [\lambda^{p-1}]} W(-)$$

V) Main thm:

① $\mathrm{Aff}(S^1) \cong B\,\mathrm{Fix}$ $\rightsquigarrow$ get a filtration on $\mathrm{Aff}(S^1)$ from above

② The filtration on $\mathrm{Aff}(S^1)$ induces the HKR filtration

on $HH(B/A)$ via $HH(B/A) \simeq R\Gamma(\underline{\mathrm{Map}}(\mathrm{Aff}(S^1), X), \mathcal{O})$

③ Analog with S^1-fixed points to get dR cohomology

Key calculations :

(1) $$C^*(S^1; k) \simeq C^*(BFix, \theta)$$

(2) $$C^*(BKer; \theta) \simeq k \oplus k[-1]$$ $\Leftarrow$ square zero extension